Statement of Research Interests

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Preface

I am a noncommutative ring theorist whose primary research interest has been homological results involving graded algebras. For example, I have studied the Koszul and $\mathscr{K}_2$ properties (see definitions below). My research has also touched algebraic topology, algebraic geometry, and representation theory.

Let $\mathbb{K}$ be a field and A be a graded $\mathbb{K}$-algebra generated by A_1 such that $A_0 = \mathbb{K}$. Let $A_+ := \bigoplus_{n>0} A_n$ and view $\mathbb{K}$ as the module ${}_A\mathbb{K} := A/A_+$. Working in the graded category, we form the vector spaces $\mathrm{E}^{i,j}(A) := \mathrm{Ext}_A^{i,j}(\mathbb{K},\mathbb{K})$. The cup product on cohomology makes $\mathrm{E}(A)$ into a bigraded algebra, called the Yoneda algebra.

This definition was first introduced by Priddy [11]:

Definition 1. The algebra A is **Koszul** if $\mathrm{E}(A)$ is generated as a $\mathbb{K}$-algebra by $\mathrm{E}^1(A)$.

The strength of Koszul is seen from the fact that there are any number of equivalent definitions given in very different forms, such as (1) homological purity, (2) distributive lattices, and (3) canonical resolutions. It is a corollary that Koszul algebras must be quadratic. The study of Koszul algebras has a deep and interesting history. A wonderful reference on this topic is the book by Polishchuk and Positselski [10]. (All the classical results involving Koszul algebras I will quote may be found there.)

In order to study questions arising from deformation theory, Cassidy and Shelton made the following generalization of Koszul [6]:

Definition 2. An algebra A is $\mathscr{K}_2$ if $\mathrm{E}(A)$ is generated as a $\mathbb{K}$-algbera by its first two cohomological degrees, $\mathrm{E}^1(A)$ and $\mathrm{E}^2(A)$.

The n-Koszul algebras are exactly the $\mathscr{K}_2$ algebras whose relations are generated in degree n [7]. (Koszul algebras are 2-Koszul.) However, unlike n-Koszul algebras, $\mathscr{K}_2$ algebras can have relations generated in more than one degree. For example, while a commutative algebra with cubic relations has no hope of being 3-Koszul (because the commutativity relations are quadratic), such algebras could be $\mathscr{K}_2$.

A survey of my results

Koszulity as a topological condition (results from [4]) Brad Shelton (my PhD advisor), Thomas Cassidy, and I investigated the Koszulity of algebras obtained from regular CW complexes. Given a regular, pure CW complex X, we define a $\mathbb{K}$-algebra $R(X)$. (We need an additional technical condition involving the $(d-1)$-cells.) Then we can relate Koszul properties of $R(X)$ with the cohomology of X:

Theorem 3. *(1) If $R(X)$ is Koszul, then $H^n(X;\mathbb{K}) = 0$ for all $0 < n < d$.*
(2) If X is a manifold or manifold with boundary and $H^k(X;\mathbb{K}) = 0$ for $0 < k < n$, then $R(X)$ is Koszul.

The proof of this theorem involves the creation of new cohomology groups which may be of interest to algebraic topologists.

More recently, in [13], Sadofsky and Shelton built off this work to show that the $R(X)$ was a topological invariant of the topological space X.

Generalizations of the Koszul condition (results from [3], [8], and [9]) Let A be as above and suppose we have an ordered basis $x_1 < \cdots < x_n$ for A_1. We can then apply a filtration on A by monomials in the x_i, giving rise to the associated graded algebra $\mathrm{gr}^F A$, which will be a monomial algebra. It was shown by Priddy [11] that if A is quadratic and $\mathrm{gr}^F A$ is quadratic (and therefore Koszul), A will be Koszul as well. (Such quadratic algebras are called **Poincaré–Birkhoff–Witt** algebras.)

In [8], I was able to extend this classic result into the world of $\mathscr{K}_2$ algebras.

Theorem 4. *A is $\mathscr{K}_2$ if $\mathrm{gr}^F A$ is $\mathscr{K}_2$ and $\dim \mathrm{E}^2(\mathrm{gr}^F A) = \dim \mathrm{E}^2(A)$.*

The condition on the dimensions of $E^2(\mathrm{gr}^F A)$ and $E^2(A)$ is equivalent to the existence of a Gröbner basis having certain nice properties. Because $\mathrm{gr}^F A$ is monomial, one can verify whether $\mathrm{gr}^F A$ is $\mathcal{K}_2$ by applying an algorithm of Cassidy and Shelton.

One nice classical property of Koszul algebras is **Koszul duality** (see [10]): If A is Koszul, then $E(E(A)) = A$. It is natural to ask if a similar property exists for $\mathcal{K}_2$ algebras. For example, Green et. al. showed that if A is n-Koszul, then $E(A)$ can be regraded so that it is Koszul, yielding a "delayed duality" $E(E(E(A))) = E(A)$.[7] A search for a similar delayed duality property is still underway. However, in [3], Cassidy, Shelton, and I show:

Theorem 5. *There exists a monomial $\mathcal{K}_2$ algebra A for which $E(A)$ is not $\mathcal{K}_2$ (in an appropriate generalized sense).*

Some of my research involves graded Ore extensions and their generalizations. In [9], I showed:

Theorem 6. *Let $B = A[z;\sigma,\delta]$ be a graded Ore extenson of A, where $\sigma : A \to A$ is a graded automorphism, and $\delta : A \to A(-1)$ is a graded σ-derivation. If B is $\mathcal{K}_2$, then A is ktwo.*

This is the converse of a theorem proved by Cassidy and Shelton in [6].

Goresky–MacPherson algebras (results from [2]) I was involved in a project with four other mathematicians involving algebras arising from algebraic geometry and representation theory. We extended the definition of Koszul to graded algebras $A = \bigoplus_{n\geq 0} A_n$ where A_0 is a semisimple ring, and a notion called Goresky–MacPherson (GM) duality which is similar to Koszul duality. If a graded algebra A is Koszul and meets some additional conditions, then the Yoneda algebra $E(A)$ will also meet those conditions and we can associate to A an algebra $\mathcal{Z}(A)$. Under these stronger conditions, the following holds:

Theorem 7. *$\mathcal{Z}(A)$ is canonically GM dual to $\mathcal{Z}(E(A))$.*

This GM duality has been observed in some examples, such as the equivariant cohomology associated to certain algebraic group actions on algebraic varieties.

Future research plans

I have developed a clear research program for the next several years.

Generalized Koszul properties for augmented algebras In order to prove Theorem 4, I extended the definition of $\mathcal{K}_2$ to the realm of augmented algebras. Rather than necessarily having a grading, these algebras merely have an augmentation map $\varepsilon : A \to \mathbb{K}$. We could make an analogous definition for Koszul algebras, calling augmented algebras having their Yoneda algebras $\mathrm{Ext}_A(\mathbb{K},\mathbb{K})$ generated in degree one $\mathcal{K}_1$. Every point in an algebraic variety gives rise to such an augmentation ε. I plan to study the $\mathcal{K}_1$ and $\mathcal{K}_2$ properties from this geometric perspective. In pursuit of this goal, I will need to further my studies of algebraic geometry.

Generalized Koszul duality As I mentioned above, one would like a version of Koszul duality for $\mathcal{K}_2$ algebras. I plan to explore this as well. As monomial algebras are the easiest examples to use in computations, I have begun exploring the structure of the Yoneda algebra for a monomial $\mathcal{K}_2$ algebra. I have also been studying the condition $E(A) \simeq E(E(E(A)))$.

Deformation theory Let $A := \mathbb{K}\langle x_1,\ldots,x_n\rangle . \langle r_1,\ldots,r_m\rangle$ and $U := \mathbb{K}\langle x_1,\ldots,x_n\rangle / \langle r_1 - w_1,\ldots,r_m - w_m\rangle$, where each r_i is homogeneous of degree d_i but $w_i \in \sum_{j=0}^{d_i-1} \mathbb{K}\langle x_1,\ldots x_n\rangle_j$. If $\mathrm{gr}\, U \simeq A$, then U is called a **Poincaré–Birkhoff–Witt deformation** of A. (For the motivating example, consider the polynomial ring $A := \mathbb{K}[x_1,\ldots x_n]$ and the universal enveloping algebra $U := U(\mathfrak{g})$ for an n-dimensional Lie algebra $\mathfrak{g}$.) In [5], Cassidy and Shelton found conditions equivalent to U being a PBW deformation of A, some of which are homological. One can view the space of all PBW deformations of a fixed algebra A as a variety X_A, which I plan to study. (For example, what does dimension information about X_A say about the algebra A?) In order to pursue this goal, I may have to learn more about algebraic group actions on varieties and equivariant cohomology.

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