Statement of Research Interests

Christopher Phan

Department of Mathematics
University of Glasgow
University Gardens
Glasgow G12 8QW
United Kingdom

Office: +44 141 330 6531
Mobile: +44 776 091 6661
http://www.maths.gla.ac.uk/~clphan/
c.phan@maths.gla.ac.uk

1. Preface

I am a noncommutative ring theorist whose primary research interest has been homological results involving graded algebras. For example, I have studied the Koszul and $\mathcal{K}_2$ properties (see definitions below). Recently, my research has branched into algebraic topology, algebraic geometry, and representation theory.

Let $\mathbb{K}$ be a field and A be a graded $\mathbb{K}$-algebra generated by A_1 such that $A_0 = \mathbb{K}$. Let $A_+ := \bigoplus_{n>0} A_n$ and view $\mathbb{K}$ as the module ${}_A\mathbb{K} := A/A_+$. Working in the graded category, we form the vector spaces $\mathrm{E}^{i,j}(A) := \mathrm{Ext}_A^{i,j}(\mathbb{K}, \mathbb{K})$. The cup product on cohomology makes $\mathrm{E}(A)$ into a bigraded algebra, called the Yoneda algebra.

This definition was first introduced by Priddy [11]:

Definition 1. The algebra A is **Koszul** if $\mathrm{E}(A)$ is generated as a $\mathbb{K}$-algebra by $\mathrm{E}^1(A)$.

The strength of Koszul is seen from the fact that there are any number of equivalent definitions given in very different forms, such as (1) homological purity, (2) distributive lattices, and (3) canonical resolutions. It is a corollary that Koszul algebras must be quadratic. The study of Koszul algebras has a deep and interesting history. A wonderful reference on this topic is the book by Polishchuk and Positselski [10]. (All the classical results involving Koszul algebras I will quote may be found there.)

In order to study questions arising from deformation theory, Cassidy and Shelton made the following generalization of Koszul [6]:

Definition 2. An algebra A is $\mathcal{K}_2$ if $\mathrm{E}(A)$ is generated as a $\mathbb{K}$-algbera by its first two cohomological degrees, $\mathrm{E}^1(A)$ and $\mathrm{E}^2(A)$.

The n-Koszul algebras are exactly the $\mathcal{K}_2$ algebras whose relations are generated in degree n [7]. (Koszul algebras are 2-Koszul.) However, unlike n-Koszul algebras, $\mathcal{K}_2$ algebras can have relations generated in more than one degree. For example, while a commutative algebra with cubic relations has no hope of being 3-Koszul (because the commutativity relations are quadratic), such algebras could be $\mathcal{K}_2$. This suggests we could explore $\mathcal{K}_2$-ness as a geometric property. For example, all complete intersections are $\mathcal{K}_2$ [6]. In [6], Cassidy and Shelton provide an algorithm that determines exactly when a noncommutative monomial algebra is $\mathcal{K}_2$. (Quadratic monomial algebras are always Koszul.)

2. Recent results

2.1. **Koszulity as a topological condition (results from [4]).** Brad Shelton (my PhD advisor), Thomas Cassidy, and I investigated the Koszulity of algebras obtained from regular CW complexes. Let X be a regular CW complex. To this regular CW complex we can associate partially-ordered sets P and $\hat{P}$. (The regularity condition is just strong enough that one can recover all topological information from P [1].) If we assume that X is a pure regular CW-complex (that is, all maximal-dimensional cells are of the same dimension), then P and $\hat{P}$ are ranked posets. To a ranked poset P we associate a graded quadratic algebra $R(P)$. (Under some circumstances, the algebras $R(\hat{P})$ are related to the algebras recently studied by Retakh, et. al. [12].) We relate certain cohomology conditions involving $R(\hat{P})$ with topological properties of X.

Suppose X is a d-dimensional pure regular CW-complex, with associated posets P and $\hat{P}$. (We need an additional technical condition involving the $(d-1)$-cells.) Then:

Theorem 3. *(1) If $R(\hat{P})$ is Koszul, then $H^n(X;\mathbb{K}) = 0$ for all $0 < n < d$.*
(2) If X is a manifold or manifold with boundary and $H^k(X;\mathbb{K}) = 0$ for $0 < k < n$, then $R(\hat{P})$ is Koszul.

The proof of this theorem involves the creation of new cohomology groups which may be of interest to algebraic topologists. It is possible to construct (non-manifold) examples in which these cohomology groups depend on the CW-decomposition of the underlying topological space.

More recently, in [13], Sadofsky and Shelton showed that the $R(\hat{P})$ was a topological invariant of the topological space X.

2.2. **Generalizations of the Koszul condition (results from [3], [8], and [9]).** Suppose

$$A = \frac{\mathbb{K}\langle x_1, \ldots, x_n\rangle}{\langle r_1, \ldots, r_m\rangle},$$

and let $\pi : \mathbb{K}\langle x_1, \ldots, x_n\rangle \twoheadrightarrow A$ be the usual projection. We can order the monomials in $\mathbb{K}\langle x_1, \ldots, x_n\rangle$ by degree-lexicographical order, and then place a filtration on A via $F_\alpha A = \pi\left(\sum_{\beta\leq\alpha}\mathbb{K}\beta\right)$. This gives rise to the associated graded algebra $\operatorname{gr}^F A$, which will be a monomial algebra. It was shown by Priddy [11] that if A is quadratic and $\operatorname{gr}^F A$ is quadratic (and therefore Koszul), A will be Koszul as well. (Such quadratic algebras are called **Poincaré–Birkhoff–Witt** algebras.)

In [8], I was able to extend this classic result into the world of $\mathcal{K}_2$ algebras.

Theorem 4. *A is $\mathcal{K}_2$ if $\operatorname{gr}^F A$ is $\mathcal{K}_2$ and $\dim \mathrm{E}^2(\operatorname{gr}^F A) = \dim \mathrm{E}^2(A)$.*

The condition on the dimensions of $\mathrm{E}^2(\operatorname{gr}^F A)$ and $\mathrm{E}^2(A)$ is equivalent to the existence of a Gröbner basis for $\ker\pi$ having certain nice properties. Because $\operatorname{gr}^F A$ is monomial, one can verify whether $\operatorname{gr}^F A$ is $\mathcal{K}_2$ by applying Cassidy and Shelton's algorithm.

One nice classical property of Koszul algebras is **Koszul duality** (see [10]): If A is Koszul, then $\mathrm{E}(\mathrm{E}(A)) = A$. It is natural to ask if a similar property exists for $\mathcal{K}_2$ algebras. For example, Green et. al. showed that if A is n-Koszul, then $\mathrm{E}(A)$ can be regraded so that it is Koszul, yielding a "delayed duality" $\mathrm{E}(\mathrm{E}(\mathrm{E}(A))) = \mathrm{E}(A)$.[7] A search for a similar delayed duality property is still underway. However, in [3], Cassidy, Shelton, and I show:

Theorem 5. *There exists a monomial $\mathcal{K}_2$ algebra A for which $\mathrm{E}(A)$ is not $\mathcal{K}_2$ (in an appropriate generalized sense).*

Some of my research involves graded Ore extensions and their generalizations. In [9], I showed:

Theorem 6. *Let $B = A[z;\sigma,\delta]$ be a graded Ore extenson of A, where $\sigma : A \to A$ is a graded automorphism, and $\delta : A \to A(-1)$ is a graded σ-derivation. If B is $\mathcal{K}_2$, then A is ktwo.*

This is the converse of a theorem proved by Cassidy and Shelton in [6].

2.3. **Goresky–MacPherson algebras (results from [2]).** I was involved in a project with four other mathematicians involving algebras arising from algebraic geometry and representation theory. We extended the definition of Koszul to graded algebras $A = \bigoplus_{n\geq 0} A_n$ where A_0 is a semisimple ring.

Definition 7. Let U be a finite-dimensional complex vector space and $S = \operatorname{Sym} U$ is the symmetric algebra. A **Goresky-MacPherson algebra** is a quadruple $\mathcal{Z} = (U, Z, \mathcal{I}, h)$, where Z is a commutative graded S-algebra, $\mathcal{I}$ is a finite set, and $h : Z \to \bigoplus_{\alpha \in \mathcal{I}} S$ is a map of graded S-algebras. If h is an isomorphism, then $\mathcal{Z}$ is a **strong** GM algebra.

We define a notion of GM duality similar to Koszul duality. If a graded algebra A is Koszul and meets some additional conditions, then the Yoneda algebra $\mathrm{E}(A)$ will also meet those conditions and we can associate to A a GM algebra $\mathcal{Z}(A)$. Under these stronger conditions, the following holds:

Theorem 8. *$\mathcal{Z}(A)$ is canonically GM dual to $\mathcal{Z}(\mathrm{E}(A))$.*

This GM duality has been observed in some examples, such as the equivariant cohomology associated to certain algebraic group actions on algebraic varieties.

3. Future research plans

I have developed a clear research program for the next two to three years.

3.1. **Algebras associated to regular CW complexes.** While the algebras $R(\hat{P})$ arising from a regular CW complex X are quadratic, there are related algebras which have mixed-degree relations. I would like to explore whether these related algebras have homological conditions such as the $\mathcal{K}_2$ property. For this exploration, I will need to learn more algebraic topology.

3.2. **Generalized Koszul properties for augmented algebras.** In order to prove Theorem 4, I extended the definition of $\mathcal{K}_2$ to the realm of augmented algebras. Rather than necessarily having a grading, these algebras merely have an augmentation map $\varepsilon : A \to \mathbb{K}$. We could make an analogous definition for Koszul algebras, calling augmented algebras having their Yoneda algebras $\mathrm{Ext}_A(\mathbb{K}, \mathbb{K})$ generated in degree one $\mathcal{K}_1$. Every point in an algebraic variety gives rise to such an augmentation ε. I plan to study the $\mathcal{K}_1$ and $\mathcal{K}_2$ properties from this geometric perspective. In pursuit of this goal, I will need to further my studies of algebraic geometry.

3.3. **Generalized Koszul duality.** As I mentioned above, one would like a version of Koszul duality for $\mathcal{K}_2$ algebras. I plan to explore this as well. As monomial algebras are the easiest examples to use in computations, I have begun exploring the structure of the Yoneda algebra for a monomial $\mathcal{K}_2$ algebra.

3.4. **GM duality.** So far, there are two known classes of examples of GM duality. I plan to use my background with noncommutative graded algebras to find other algebras.

3.5. **Deformation theory.** Let

$$A := \frac{\mathbb{K}\langle x_1, \ldots, x_n \rangle}{\langle r_1, \ldots, r_m \rangle}$$

and

$$U := \frac{\mathbb{K}\langle x_1, \ldots, x_n \rangle}{\langle r_1 - w_1, \ldots, r_m - w_m \rangle},$$

where each r_i is homogeneous of degree d_i but $w_i \in \sum_{j=0}^{d_i - 1} \mathbb{K}\langle x_1, \ldots x_n \rangle_j$. If $\operatorname{gr} U \simeq A$, then U is called a **Poincaré–Birkhoff–Witt deformation** of A. (For the motivating example, consider the polynomial ring $A := \mathbb{K}[x_1, \ldots x_n]$ and the universal enveloping algebra $U := U(\mathfrak{g})$ for an n-dimensional Lie algebra $\mathfrak{g}$.) In [5], Cassidy and Shelton found conditions equivalent to U being a PBW deformation of A, some of which are homological. One can view the space of all PBW deformations of a fixed algebra A as a variety X_A, which I plan to study. (For example, what does dimension information about X_A say about the algebra A?) In order to pursue this goal, I may have to learn more about algebraic group actions on varieties and equivariant cohomology.

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