# Noncommutative Koszul algebras from combinatorial topology

Christopher Phan

Department of Mathematics
University of Oregon

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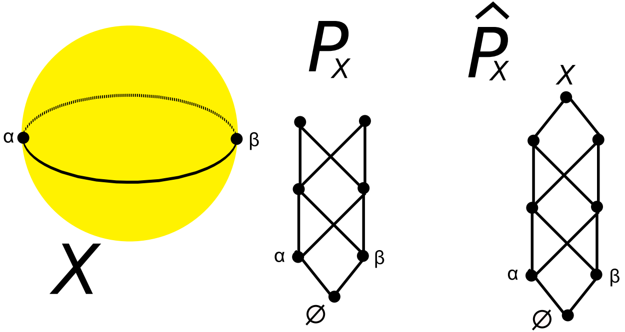
# Set-up

- $X$ is a pure, regular CW-complex of dimension $n$.
- We can associate to $X$ two ranked partially-ordered sets $P_X$ and $\hat{P}_X$. (The regularity condition means we can recover all the topological structure of $X$ from $P_X$.)

- Example

$X = S^2$.



# Splitting algebras associated to $X$

- Given a path $\pi : a_1 \xrightarrow{e_1} a_2 \xrightarrow{e_2} \cdots \xrightarrow{e_n} a_{n-1}$ in $P_X$ (resp. $\hat{P}_X$), we set
$$p_\pi(t) = (t - e_1)(t - e_2)\cdots(t - e_n)$$
where $t$ is a central indeterminate.
- $A(P_X)$ (resp. $A(\hat{P}_X)$) is the universal algebra obtained by equating coefficients of $p_\pi$ and $p'_\pi$, where $\pi, \pi'$ are paths with the same head and tail.

- Theorem (Retakh, Serconek, and Wilson)

  *If $X$ is connected by $(n-1)$-faces (mild technical hypothesis), then $A(P_X)$ and $A(\hat{P}_X)$ are quadratic algebras.*
  - **Question:** When are $A(P_X)$ are $A(\hat{P}_X)$ Koszul algebras?

- Theorem (Serconek and Wilson; Piontkovski)

  *If $X$ is the n-simplex, then $A(P_X)$ is Koszul.*

# Koszul algebras

- A connected-graded algebra $A$ is **Koszul** if $\text{Ext}_A(\mathbb{K}, \mathbb{K})$ is generated by $\text{Ext}^1_A(\mathbb{K}, \mathbb{K})$ as an algebra.
- Why do we study Koszul algebras?
  - Because the Koszul condition is equivalent to any of the following:
    - ${}_A\mathbb{K}$ has a canonical linear resolution.
    - $\text{Ext}^{i,j}_A(\mathbb{K}, \mathbb{K}) = 0$ unless $i = j$.
    - $A$ is quadratic and $\text{Ext}_A(\mathbb{K}, \mathbb{K})$ is quadratic with explicit relations.
    - $A$ is associated to a certain distributive lattice.
    - Several others.
  - Koszul algebras are extremely common.

# Main results

- We assume $X$ meets the mild technical hypothesis.

- Theorem (Cassidy, P., and Shelton)
  *$A(P_X)$ is always Koszul.*

- Theorem (Cassidy, P., and Shelton)
  1. *If $X$ is 2-dimensional and $H^1(X, \mathbb{K}) = 0$, then $A(\hat{P}_X)$ is Koszul.*
  2. *If $X$ is a manifold (possibly with boundary) and $H^n(X, \mathbb{K}) = 0$ for $0 < k < n$, then $A(\hat{P}_X)$ is Koszul.*
  3. *$A(\hat{P}_X)$ is not always Koszul.*

- Part 1 extends a theorem of Retakh, Serconek, and Wilson in `arXiv:0810.1241`.

# Methodology

- There exists finite-dimensional, quadratic algebras $R(P_X)$ and $R(\hat{P}_X)$ such that:
  - $R(P_X)$ is Koszul implies $A(P_X)$ is Koszul.
  - $R(\hat{P}_X)$ is Koszul implies $A(\hat{P}_X)$ is Koszul
- We introduce, for every $0 \leq k \leq m \leq n$, new combinatorial cohomology groups $H_X(m, k)$ which generalize $H^m(X)$. (In particular, $H^m(X) = H_X(m, 0)$.)
- Both theorems above are corollaries to:

**Theorem (Cassidy, P., and Shelton)**

*$R(\hat{P}_X)$ is Koszul if and only if $H_X(m, k) = 0$ for all $0 \leq k < m < d$.*

Preprint: `arXiv:0811.3450`

# Examples

- Since $H^1(\mathbb{RP}^2; \mathbb{K}) = 0$ if and only if $\mathrm{char}\, K \neq 2$, we can create algebras whose Koszulity depends on the characteristic of the base field.
- The poset below has its $A(P)$ *not* Koszul, which you can determine from the geometry. (The underlying manifold is contractible.)