$\mathcal{K}_2$ properties for augmented algebras

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Motivation: Poincaré–Birkhoff–Witt algebras

- Let $A = \mathbb{T}(V)/I$ be a connected-graded $\mathbb{K}$-algebra, where $\mathbb{K}$ is a field.
- Fix an ordered basis $x_1 < x_2 < \cdots < x_n$ for V.
- This makes the set of monomials $\mathcal{M}$ of $\mathbb{T}(V)$ a totally-ordered monoid by degree-lexicographical order.
- Define a filtration F on A by $\mathcal{M}$ via $F_\alpha := \sum_{\beta \leq \alpha} \beta + I$.

Definition

Suppose A is a quadratic algebra. Then A is a **Poincaré–Birkhoff–Witt algebra** if $\operatorname{gr}^F A$ is a quadratic algebra.

Theorem (Priddy 1970)

PBW algebras are Koszul.

$\mathcal{K}_2$ algebras

- We'll use the notation $\mathrm{E}(A) := \mathrm{Ext}_A(\mathbb{K}, \mathbb{K})$.
- Recall, one definition of Koszulity is that $\mathrm{E}(A)$ is generated by $\mathrm{E}^1(A)$.

Definition (Cassidy and Shelton 2007)

A is a $\mathcal{K}_2$ algebra if $\mathrm{E}(A)$ is generated by $\mathrm{E}^1(A)$ and $\mathrm{E}^2(A)$.

- This allows for algebras that have homogeneous relations in different degrees.

Problem

Find a generalization of Priddy's theorem for $\mathcal{K}_2$ *algebras.*

Essential Gröbner bases

Definition (Cassidy and Shelton 2007)

Let $I' := I \otimes V + V \otimes I$. Then $x \in I$ is an **essential relation** if $x \notin I'$.

Definition

A set $\mathcal{G} \subset I$ of essential relations is an **essential generating set** if $\mathcal{G}$ generates I but no proper subset does.

- If a quadratic algebra A is a PBW algebra, then I has a Gröbner basis that is also an essential generating set.

Definition

We'll call a Gröbner basis that is also an essential generating set an **essential Gröbner basis**.

$\mathcal{K}_2$ analogue of Priddy's theorem

Theorem (P. 2007)

Let $A := \mathbb{T}(V)/I$ *be a connected-graded algebra. If* I *has an essential Gröbner basis and* $\mathrm{gr}^F A$ *is* $\mathcal{K}_2$*, then* A *is* $\mathcal{K}_2$*.*

- The algebra $\mathrm{gr}^F A$ is monomial. Cassidy and Shelton have provided an algorithm for determining whether a monomial algebra is $\mathcal{K}_2$.
- In the quadratic case, $\mathrm{gr}^F A$ being quadratic assures us it is Koszul, as monomial quadratic algebras are always Koszul.

Examples

Example

Let

$$A := \frac{\mathbb{K}\langle x, y\rangle}{\langle yx - xy, y^3 + x^2 y\rangle}.$$

Under the order $x < y$, the essential generating set $\{yx - xy, y^3 + x^2 y\}$ is a Gröbner basis for the ideal of relations, and

$$\operatorname{gr}^F A \simeq \frac{\mathbb{K}\langle x, y\rangle}{\langle yx, y^3\rangle}$$

can be shown to be $\mathcal{K}_2$. So, A is a $\mathcal{K}_2$ algebra.

Examples

Example

Let

$$A := \frac{\mathbb{K}\langle x, y\rangle}{\langle xy - x^2, yx, y^3\rangle}.$$

Under the order $x < y$, we have

$$\operatorname{gr}^F A \simeq \frac{\mathbb{K}\langle x, y\rangle}{\langle xy, yx, x^3, y^3\rangle},$$

which can be shown to be $\mathcal{K}_2$. However, A is not a $\mathcal{K}_2$ algebra. The generating set $\{xy - x^2, yx, y^3\}$ is not a Gröbner basis, and no essential Gröbner basis exists.

Examples (Converse false!)

Example

Let

$$A := \frac{\mathbb{K}\langle x, y, z\rangle}{\langle x^2y - x^3, yz^2 - yx^2, x^3z - x^4\rangle},$$

which is a $\mathcal{K}_2$ algebra. Under the order $x < y < z$,

$$\operatorname{gr}^F A \simeq \frac{\mathbb{K}\langle x, y, z\rangle}{\langle x^2y, yz^2, x^3z\rangle},$$

and $\{x^2y - x^3, yz^2 - yx^2, x^3z - x^4\}$ is an essential Gröbner basis. However, $\operatorname{gr}^F A$ is *not* $\mathcal{K}_2$.

The ungraded case

- Now, suppose A is augmented algebra, i.e. there is a surjective map $\varepsilon : A \twoheadrightarrow \mathbb{K}$. Denote $A_+ := \ker \varepsilon$.
- Now $\mathrm{E}(A) := \mathrm{Ext}_A(\mathbb{K}, \mathbb{K})$ (in the ungraded category).
- When A is also graded, we'll use $\mathrm{E}_{\mathrm{Gr}}(A)$ to mean the Yoneda algebra in the graded category.
- When A is also graded, $\mathrm{E}^n(A) = \mathrm{E}^n_{\mathrm{Gr}}(A)$ if and only if both are finite dimensional.

The ungraded case

- Suppose A is filtered by a totally-ordered monoid $\mathcal{M}$ such that
  1. $\bigcup_\alpha F_\alpha A = A$;
  2. $F_\alpha A = \mathbb{K} \oplus F_\alpha A_+$, where $F_\alpha A_+ := F_\alpha A \cap A_+$;
  3. $F_e A = \mathbb{K}$ and $F_\alpha A_+ \neq 0$ when $\alpha > e$; and
  4. $\dim F_\alpha A / F_{s(\alpha,-1)} A < \infty$ for all $\alpha > e$.
- This will induce a filtration on $\mathrm{Cob}^\bullet(A) := \mathrm{Hom}_{\mathbb{K}}(A_+^\bullet, \mathbb{K})$ (in the ungraded category).
- We can push this filtration to $\mathrm{E}(A)$.

Theorem (P. 2007)

There is a bigraded (with respect to the cohomological and $\mathcal{M}$ gradings) algebra monomorphism

$$\Lambda : \mathrm{gr}^F \mathrm{E}(A) \hookrightarrow \mathrm{E}_{\mathrm{Gr}}(\mathrm{gr}^F A).$$

Basic idea of the proof

$$\begin{tikzcd}
 & F_\alpha \operatorname{Cob}^n(A) \cap \ker \partial \arrow[dl, "\eta_\infty"'] \arrow[dr, "\eta_1"] & \\
(\operatorname{gr}^F \mathrm{E}(A))^{n,\alpha} \arrow[rr, dashed, "\Lambda^{n,\alpha}"] & & \mathrm{E}^{n,\alpha}_{\mathrm{Gr}}(\operatorname{gr}^F A)
\end{tikzcd}$$

- The Yoneda algebra $\mathrm{E}(A)$ is the cohomology of the cobar complex $\operatorname{Cob}^\bullet(A) := \operatorname{Hom}(A_+^{\otimes\bullet}, \mathbb{K})$.
- The cobar complex is filtered by $\mathcal{M}$.
- The maps η_1 and η_∞ appear in the construction of a spectral sequence—although we don't use this spectral sequence.
- Then Λ can be found so that the diagram commutes.

The ungraded case

- We generalize the definition of $\mathcal{K}_2$ to the ungraded augmented case (i.e. an augmented algebra is $\mathcal{K}_2$ when $\mathsf{E}(A)$ is generated by $\mathsf{E}^1(A)$ and $\mathsf{E}^2(A)$).
- In the connected-graded case:
  - If the algebra is $\mathcal{K}_2$ in the original sense, $\mathsf{E}_{\mathrm{Gr}}(A) = \mathsf{E}(A)$.
  - The graded and ungraded definitions of $\mathcal{K}_2$ coincide.
  - Λ_1 is always surjective
  - Having an essential Gröbner basis corresponds exactly to the surjectivity of Λ_2.

Theorem (P. 2007)

If $\mathsf{E}_{\mathrm{Gr}}(\mathrm{gr}^F A)$ *is generated in the first two cohomological degrees and* Λ_1, Λ_2 *are surjective, then* A *is* $\mathcal{K}_2$.

Example

Example

Let

$$A := \mathbb{K}[x, y]/\left\langle x^3 - p \right\rangle,$$

where p is a homogeneous quadratic polynomial, filtered by degree, with the augmentation $\varepsilon(x) = \varepsilon(y) = 0$. Then

$$\operatorname{gr} A \simeq \mathbb{K}[x, y]/\left\langle x^3 \right\rangle$$

is $\mathcal{K}_2$. We can show that Λ_1 and Λ_2 are surjective, and use this to show A is $\mathcal{K}_2$.

Possible future directions

- Are there geometric implications for $\mathcal{K}_2$-ness of augmented algebras?
- What kinds of ungraded algebras are $\mathcal{K}_1$ (i.e. have $\mathrm{E}(A)$ generated by $\mathrm{E}^1(A)$)?
- Might help us better understand when $\mathrm{E}(\mathrm{E}(A))$ is $\mathcal{K}_2$.

Preprint at `arXiv:0711.3480v1` [math.RA].